Paper ID: MS-41

An Iteration Method for Near-tip singular fields of a crack in a Power- law Hardening Material Using Asymptotic Analysis

Md. Arefin Kowser[1], Md. Mahiuddin[2]
[1]Department of Mechanical Engineering, Dhaka University of Engineering & Technology, Gazipur,
[2]Department of Mechanical Engineering, Dhaka University of Engineering & Technology, Gazipur
E-mail: nadimduet@yahoo.com

Abstract

In this paper an asymptotic analysis of the near-tip stress and displacement fields of a crack in a power-law hardening material has been presented. Singular exponent λ is investigated for the determination of singular stress and displacement fields for the crack. Symmetric loading condition is applied in this analysis. To determine the singular exponent,λ strain compatibility equation has been derived where 4^{th} order differential equation has been solved using Range-Kutta method. The solution technique of non-linear compatibility equation, the calculation results are presented as well. All results are calculated with the assumption of plane strain condition. The solution obtained for λ is compared with the results of other researcher solved for crack problem.

Keywords: *Asymptotic analysis, singular stress, singularity, compatibility equation, Range-Kutta method, Non-linear solution, crack.*

1. Introduction

In recent years, a considerable amount of work has been done in studying crack growth behavior in power-law hardening materials [1-4]. The importance of these studies stems from the fact that the crack growth behavior in the material may significantly affect the integrity of the structure made of that material. The basic approach used in characterizing the particulate material is based on linear elastic or non linear fracture mechanics. According to the theories, crack growth behavior is controlled by the local stress/strain near the crack tip. Therefore, the values of local stress/strain near the crack tip must be determined. When crack occurs, the high stress at the crack tip will induce high damage near the crack tip region. The high damage zone at the crack tip is defined as the failure process zone, which is a key parameter in fracture mechanics [14-16]. Experimental data reveal that when the local strain reaches a critical value, small voids are generated in the failure process zone. The heterogeneity of the microstructure plays a key role for local damage and strain distributions near the crack tip. Experimental results indicate that the high strain field is localized within 1 mm of the crack tip. Also, the time - dependent damage initiation and evolution processes are contributing factors to the non-steady crack growth in this material [17-19]. Recently the interfacial crack mechanism expected much interest, largely due to the existence of interfacial fracture in lots of advanced materials such as polycrystalline alloys, structural ceramics, and composites. Early research work on this field can be found in Williams, England, Erdogan, Rice and Sih and so on [5-10]. As for the interface cracks of the elastic-plastic materials, a significant progress was made by Shih and Asaro in 1988 [11]. Through the detail full-field computational investigations by the finite element method, they found that the near tip fields of crack on biomaterial interfaces have the nearly separable form solutions of the HRR type in an annular region within the plastic zone. Wang presented an exact asymptotic analysis for a crack lying on the interface of an elastic-plastic material and linear elastic material. A separable singular stress field of the HRR type has been found in the plastic angular zone around crack tip [12].

A crack in a far field which is tensile in a direction perpendicular to it is considered. A solution to compatibility equation is sought in the immediate vicinity of the crack tip where the stresses are large, and, on the basis of the theory which has been laid down, are unbounded as the crack tip approached [13]. An asymptotic expansion of the solution is attempted in the form,

$$\phi = r^s \, \widetilde{\phi_1}(\theta) + r^t \, \widetilde{\phi_2}(\theta) + \ldots\ldots.. \quad (1)$$

Where, if the first term is to be singled out as the dominant one, s < t etc, our search will be restricted to only the dominant term in such an expansion,

$$\phi = Ar^{\lambda+1}\, \tilde{\phi}\,(\theta) \tag{2}$$

Where the amplitude A will permit us to adjust the amplitude of $\tilde{\phi}$ in some arbitrary, yet appropriate manner.

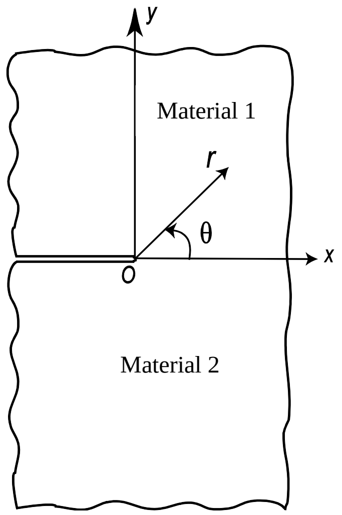

Fig.1: Conventions at crack tip

2. Mathematical Equations

Stress and strain relationships

The generalized dimensionless relationship between strain and stress governed by a power-law form and J_2– deformation theory is

$$\varepsilon_{ij} = (1+\nu)s_{ij} + \frac{(1-2\nu)}{3}\sigma_{kk}\delta_{ij} + \frac{3}{2}\alpha\sigma_e^{n-1}s_{ij} \tag{3}$$

$$\varepsilon_{ij} = (1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij} + \frac{3}{2}\alpha\sigma_e^{n-1}s_{ij} \tag{4}$$

$\sigma_{kk} = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}$; For plane strain condition: $\varepsilon_{zz} = 0;\ \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$

Where, $\varepsilon_{ij} = \dfrac{\bar{\varepsilon}_{ij}}{\varepsilon_y}$, $\varepsilon_y = \sigma_y/E$, $s_{ij} = \dfrac{\bar{s}_{ij}}{\sigma_y}$ and $\sigma_{ij} = \dfrac{\bar{\sigma}_{ij}}{\sigma_y}$ (5)

$$\sigma_e = \sqrt{\frac{3}{2}s_{ij}s_{ij}} \tag{6}$$

And, $s_{ij} = \sigma_{ij} - \dfrac{1}{3}\sigma_{kk}\delta_{ij}$ (7)

It is noted that the barred quantities in Eq. (5) are the non-normalized field variables, $(\sigma_y, \varepsilon_y)$ is a reference point for the uniaxial stress -strain curve, $\bar{\sigma}_{ij}, \bar{\varepsilon}_{ij}$ *and* $\bar{s}_{ij}$ are the stress, strain and deviatoric stress component along ij direction, where i , j are used for subscript indicates x, y or r, θ (ij denotes rr , $\theta\theta$ and $r\theta$). E is the initial slope of the stress-strain curve called Young's modulus, δ_{ij} is the two dimensional Kronecker delta symbol, ν is the Poisson ratio, α and n are hardening coefficient and hardening exponent, respectively. σ_e is the effective stress and s_{ij} is the stress deviator. Stress quantities normalized by yield stress and strain quantities are normalized by corresponding yield strain.

Equilibrium

In problems, the equilibrium equations are automatically satisfied for all stresses derived from the Airy stress function ϕ when the stresses are defined in the following manner,

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \tag{8}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} \tag{9}$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} \tag{10}$$

Where ϕ is called the Airy stress function.

The dimensionless stress function ϕ and coordinate r are given in terms of dimensional quantities (barred) by

$\phi = \dfrac{\overline{\phi}}{\sigma_y L^2}$ and $r = \dfrac{\overline{r}}{L}$,where L is the characteristic length (for example half length of the crack).

Compatibility and strain-displacement equations

Using small deformation theory, the compatibility equation in terms of strain components is written,

$$-\frac{1}{r}\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{2}{r}\frac{\partial \varepsilon_{\theta\theta}}{\partial r} + \frac{\partial^2 \varepsilon_{\theta\theta}}{\partial r^2} - \frac{2}{r^2}\frac{\partial \varepsilon_{r\theta}}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2} - \frac{2}{r}\frac{\partial^2 \varepsilon_{r\theta}}{\partial r \partial \theta} = 0 \tag{11}$$

Plane strain condition

$$\varepsilon_{zz} = 0;\ or\ \ \sigma_{zz} = \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)$$

$$\varepsilon_{rr} = \left(1+\nu\right)\left\{\left(1-\nu\right)\sigma_{rr} - \nu\sigma_{\theta\theta}\right\} + \frac{3}{2}\alpha\sigma_e^{n-1}s_{rr} \tag{12}$$

$$\varepsilon_{\theta\theta} = \left(1+\nu\right)\left\{\left(1-\nu\right)\sigma_{\theta\theta} - \nu\sigma_{rr}\right\} + \frac{3}{2}\alpha\sigma_e^{n-1}s_{\theta\theta} \tag{13}$$

$$\varepsilon_{r\theta} = \left(1+\nu\right)\sigma_{r\theta} + \frac{3}{2}\alpha\sigma_e^{n-1}s_{r\theta} \tag{14}$$

In this approximation, the elastic strains in compared with the plastic strains are small and can be neglected in the asymptotic analysis. Hence according to the plastic deformation theory the three dimensional stress-strain relations take the form,

$$\varepsilon_{ij} = \frac{3}{2}\alpha\sigma_e^{n-1}s_{ij} \tag{15}$$

The small deformation strain-displacement relations are as follows,

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \tag{16}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}u_r + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \tag{17}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \tag{18}$$

Boundary conditions

$$\left(\sigma_{\theta\theta}\right)_{\theta=0}=0 \quad \text{and} \quad \left(\sigma_{\theta\theta}\right)_{\theta=\pi}=0$$
$$\left(\sigma_{r\theta}\right)_{\theta=0}=0 \qquad \left(\sigma_{r\theta}\right)_{\theta=\pi}=0 \tag{19}$$

Governing Differential Equation

Assumed, $\phi = Ar^{\lambda+1}\tilde{\phi}$

Compatibility equation becomes in the form of,

$$C\times\frac{\partial^4\tilde{\phi}}{\partial\theta^4}=-B\left(\frac{\partial^3\tilde{\phi}}{\partial\theta^3},\,\frac{\partial^2\tilde{\phi}}{\partial\theta^2},\,\frac{\partial\tilde{\phi}}{\partial\theta},\,\tilde{\phi},\,n,\,\lambda\right) \tag{20}$$

Where B and C are derived using Mathematica software. Equation (20) is the fourth-order ordinary differential equation of $\tilde{\phi}_1^k$. The governing differential equation and boundary conditions define an Eigen value problem. A fourth-order Runge-Kutta method and the shooting method were used to solve the problem. The Mechanical properties of power-law hardening materials are given in Table 1. E is Young's modulus, ν is the Poisson's ratio.

Table 1. Mechanical Properties of cracked materials

Properties	E[GPa]	ν	σ_y [MPa]	n	α	p_0 [MPa]
Material I	108	0.33	30	3	10.1	130
Material II	108	0.33	30	3	10.1	130

Stress calculation

$$\sigma_{rr}=Ar^{\lambda-1}\left[\left(\lambda+1\right)\tilde{\phi}+\frac{\partial^2\tilde{\phi}}{\partial\theta^2}\right] \tag{21}$$

$$\sigma_{\theta\theta}=Ar^{\lambda-1}\left(\lambda+1\right)\lambda\tilde{\phi}. \tag{22}$$

$$\sigma_{r\theta}=-Ar^{\lambda-1}\lambda\frac{\partial\tilde{\phi}}{\partial\theta} \tag{23}$$

$$\sigma_e=Ar^{\lambda-1}\times2^{1-n}\times3^{\frac{n-1}{2}}\left\{\left(\lambda^2-1\right)^2\tilde{\phi}^2-2\left(\lambda^2-1\right)\tilde{\phi}\left(\frac{\partial^2\tilde{\phi}}{\partial\theta^2}\right)+\left(\frac{\partial^2\tilde{\phi}}{\partial\theta^2}\right)^2+4\lambda^2\left(\frac{\partial\tilde{\phi}}{\partial\theta}\right)^2\right\}^{\frac{1}{2}} \tag{24}$$

Displacement Calculation

$$u_{rp}=2^{-1-n}3^{\frac{n+1}{2}}\alpha A^n\frac{r^{n\lambda-n+1}}{\left(n\lambda-n+1\right)}\left(\left(1-\lambda^2\right)\tilde{\phi}+\frac{\partial^2\tilde{\phi}}{\partial\theta^2}\right)$$
$$\times\left\{\left(\lambda^2-1\right)^2\tilde{\phi}^2-2\left(\lambda^2-1\right)\tilde{\phi}\left(\frac{\partial^2\tilde{\phi}}{\partial\theta^2}\right)+\left(\frac{\partial^2\tilde{\phi}}{\partial\theta^2}\right)^2+4\lambda^2\left(\frac{\partial\tilde{\phi}}{\partial\theta}\right)^2\right\}^{\frac{n-1}{2}} \tag{25}$$

$$
\begin{aligned}
u_{\theta p} &= \frac{-2^{-1-n}3^{\frac{n+1}{2}}\alpha A^n r^{n\lambda-n+1}}{\{n(\lambda-1)\}(n\lambda-n+1)}\left\{(\lambda^2-1)^2\tilde{\phi}^2-2(\lambda^2-1)\tilde{\phi}\left(\frac{\partial^2\tilde{\phi}}{\partial\theta^2}\right)+\left(\frac{\partial^2\tilde{\phi}}{\partial\theta^2}\right)^2+4\lambda^2\left(\frac{\partial\tilde{\phi}}{\partial\theta}\right)^2\right\}^{\frac{n-3}{2}} \\
&\times\Bigg[4\lambda^2\{(4n(\lambda-1)-\lambda+4)\lambda+1\}\frac{\partial^3\tilde{\phi}}{\partial\theta^3}+\{(7n-4)\lambda^2-4(n-1)\lambda+n\}\left(\frac{\partial^2\tilde{\phi}}{\partial\theta^2}\right)^2\frac{\partial\tilde{\phi}}{\partial\theta} \\
&+\left\{4\lambda^2\left(\frac{\partial\tilde{\phi}}{\partial\theta}\right)^2+n\left(\frac{\partial^2\tilde{\phi}}{\partial\theta^2}\right)^2\right\}\frac{\partial^3\tilde{\phi}}{\partial\theta^3}+2(\lambda^2-1)\tilde{\phi}\left(\frac{\partial^2\tilde{\phi}}{\partial\theta^2}\right)\left\{-\{(5n-2)\lambda^2-4(n-1)\lambda+n\}\frac{\partial\tilde{\phi}}{\partial\theta}-n\frac{\partial^3\tilde{\phi}}{\partial\theta^3}\right\} \\
&+(\lambda^2-1)^2\tilde{\phi}^2\left\{(3n\lambda^2-4n\lambda+4\lambda+n)\frac{\partial\tilde{\phi}}{\partial\theta}+n\frac{\partial^3\tilde{\phi}}{\partial\theta^3}\right\}\Bigg]
\end{aligned}
\tag{26}
$$

To satisfy the boundary conditions for the crack case

$\frac{\partial\tilde{\phi}}{\partial\theta}=0$ And $\frac{\partial^3\tilde{\phi}}{\partial\theta^3}=0$, where other unknowns $\tilde{\phi}$, $\frac{\partial^2\tilde{\phi}}{\partial\theta^2}$, λ also calculated.

Assumed at $\theta=0$, $\tilde{\phi}=1.0$, $\frac{\partial^2\tilde{\phi}}{\partial\theta^2}$, λ and after integration $\tilde{\phi}=\tilde{\phi}^i$, $\frac{\partial\tilde{\phi}}{\partial\theta}=\left(\frac{\partial\tilde{\phi}}{\partial\theta}\right)^i$, $\frac{\partial^2\tilde{\phi}}{\partial\theta^2}=\left(\frac{\partial^2\tilde{\phi}}{\partial\theta^2}\right)^i$ and $\frac{\partial^3\tilde{\phi}}{\partial\theta^3}=\left(\frac{\partial^3\tilde{\phi}}{\partial\theta^3}\right)^i$ where the error are calculated as,

$$error1=(\lambda+1)\lambda\tilde{\phi}^i \tag{27}$$

$$error2=-\lambda\left(\frac{\partial\tilde{\phi}}{\partial\theta}\right)^i \tag{28}$$

Finally the error value is calculated by, $error=\sqrt{(error1)^2+(error2)^2}$ (29)

The final solution is obtained from the minimum error region and calculated numerically the solution where the minimum error occurs. The solution point finding techniques are shown in Fig. 2 and Fig. 3.

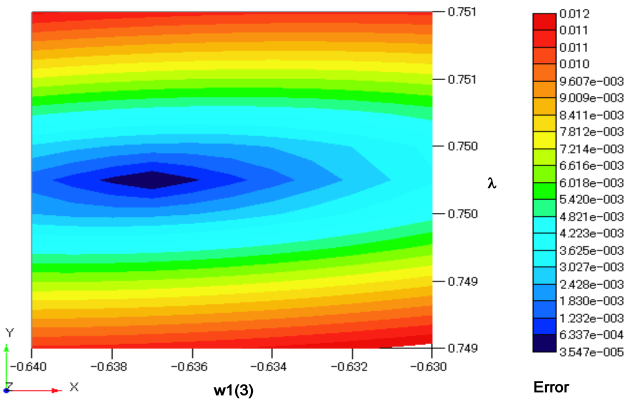

Fig. 2: Identify the minimum error region for $w1(3)$, λ

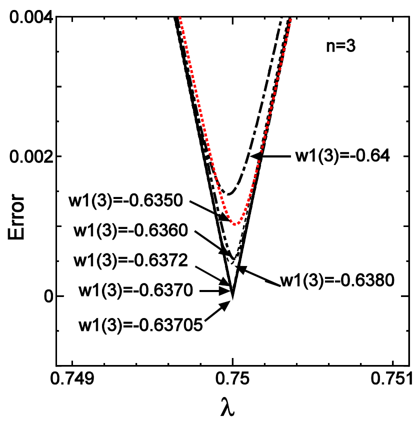

Fig. 3: Point which shows the minimum error

3. Results and Conclusion

Fig.3 finally shows the solution point (point which shows the minimum error is the solution point). The value of $w1(3)=-0.63705\pm5\times10^{-5}$ and $\lambda=0.75\pm1\times10^{-5}$ obtained for the minimum error.

Characteristics of singular stress field near the crack-tip are investigated numerically from the solution of differential equation governing the stress function obtained from compatibility equation. Fig. 4 shows the angular variation of normalized stresses near the tip of a crack for plane strain condition under the symmetric loading condition. The angular variation of normalized displacements for plane strain is shown in Fig. 5.

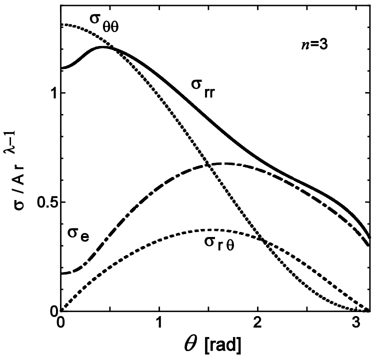

Fig. 4: θ - variations for normalized stresses at crack tip for plane strain condition

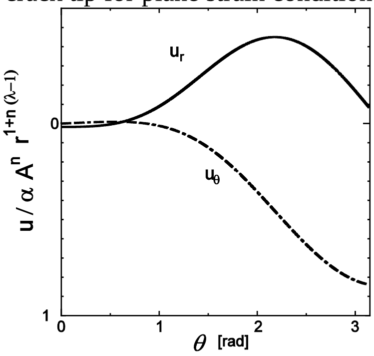

Fig.5: θ - variations for normalized displacement at n = 3

The stress fields can be expressed by $r^{\lambda-1}$ and displacement is proportional to $r^{1-n(\lambda-1)}$.Where r is the distance from the crack tip and λ is the Eigen value called singular exponent. For the continuity of displacements requires $\lambda > 0$ and from the stress fields' unbounded stress occur when $\lambda < 1$, a value of λ such that $0 < \lambda < 1$ will give unbounded stress near the tip of the crack.

For presenting the results, it is necessary to give an idea about the accuracy of the numerical technique and the reliability of the program used. Here, ten thousand segments are taken along the angular edges, and fourth-order Runge-Kutta method is used for the solution of differential equation. An agreement has also been obtained with the solutions investigated by Hutchinson is presented in Table 2 and the value of λ is 0.75 measured for the power-law hardening exponent n = 3 for the plane strain condition under symmetric loading for the crack. Stress and displacement fields for n = 5, 7 and n = 13 are also determined which are not presented here. Which means presented iteration method is suitable for the determination of near tip singular fields for crack of power-law materials of different co-efficient. In the future study, the solution will be presented the effect of the addition of another term with the dominating first term iteratively.

Table 2. Comparisons of results of nonlinear crack (plane strain condition)

Power-law hardening exponent, n	$\tilde{\phi}$	$\frac{\partial^2 \tilde{\phi}}{\partial \theta^2}$	Error	λ	Error	by J.W.Hutchinson
3	1.0	-0.63705	$\pm 5 \times 10^{-5}$	0.75	$\pm 1 \times 10^{-5}$	0.75

4. References

[1] Liu, C.T. and Smith, C.W., "Near Tip Behavior in a Particulate Composite Material- An Experimental and Analytical Investigation," Proc. Of 2n International Conf. on Computational Engineering Science, Aug. 1991

[2] Smith, C.W. and Liu, C.T.," Effects of Near-Tip Behavior of Particulate Composites on Classical Concepts," Journal of Composite Engineering, Vol.1, No 4, 1991

[3] Liu, C.T. and Ravi-Chandar,K.," Local Fracture and Crack Growth Behavior in a Particulate Composite Material," Journal of reinforced Plastics and Composites, Vol.l5/No.2 1996.

[4] Liu, CT.," The Effect of Micro Damage on Time-Dependent Crack Growth in a Composite Solid Propellant," Journal of Mechanics of Time Dependent Materials, Vol.1, 1997.

[5] Williams ML. Bull Seismol,"The stress around a fault or crack in dissimilar media." Soc America, Vol. 49, pp. 199-204, 1959

[6] England AH."A Crack Between Dissimilar Media." J Appl Mech, Vol. 32, pp. 400-402, 1965

[7] Erdogan F., "Stress Distribution in a Xonhomogeneous Elastic Plane with Cracks." J. Appl Mech, Vol. 30, pp. 232-236, 1963

[8] Rice JR and Sih GC. , "The Bending of Plates of Dissimilar Materials With Cracks," J Appl Mech, Vol. 32, pp. 418-443, 1965

[9] Hutchinson JW, Mear M and Rice JR., "Crack Paralleling an Interface Between Dissimilar Materials.", J Appl Mech, Vol. 54, pp. 828-832, 1987

[10] Rice JR., "Elastic fracture mechanics concepts for interfacial cracks." J A. M, Vol. 55, pp.98-103,1988

[11] Shih CF and Asaro RJ., "Elastic-plastic analysis of cracks on bimaterial interfaces: Part I—small scale yielding." J Appl Mech, Vol. 55, pp. 299-316, 1988

[12] Wang TC., "Higher-order analysis of near-tip fields around an interfacial crack between two dissimilar power-law hardening materials" Engng Fract Mech, Vol. 37, pp. 525-538, 1990

[13] JW., "Plastic stress and strain fields at a crack tip", J Mech Phys Solids, Vol 16, pp. 13-31, 1968

[14] Shih CF, Asaro RJ., "Elastic plastic analysis of cracks on bimaterial interfaces: Part II -Structure of small-scale yielding fields.", J Appl Mech, Vol. 56, pp. 763-779, 1989

[15] Shih CF, Asaro RJ, O'Dowd NP., "Elastic-plastic analysis of cracks on bimaterial interfaces: Part lII—large-scale yielding." J Appl Mech, Vol. 58, pp. 450-463, 1991

[16] Shih CF, Asaro RJ., "Elastic-plastic and asymptotic fields of interface cracks." Int J Fract, Vol. 42, pp. 101-116, 1990

[17] Aravas N, Sharma SM., "An elastoplastic analysis of the interface crack with contact zone." J Mech Phys Solids, Vol. 39, pp. 311-344, 1991

[18] Shih CF., "Cracks on bimaterial interfaces: elasticity and plasticity aspects." Mater. Sci Engng, Vol. A 143, pp. 77-90, 1991

[19] Shih CF., "Small scale yielding analysis of mixed mode plane strain crack problems." Fracture analysis. ASTM STP, 560, pp. 187-210, 1974